Curvature dimension inequalities and subelliptic heat kernel gradient bounds on contact manifolds

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Abstract

We extend to any contact manifold the curvature dimension inequality that was introduced by Baudoin and Garofalo in [2]. In particular, the Sasakian condition is no longer assumed which leads to the appearance of a new term in the curvature dimension inequality. This new curvature dimension condition is then used to study:

- The stochastic completeness of the heat semigroup associated to the contact sub-Laplacian;
- Geometric conditions ensuring the compactness of the underlying manifold (Bonnet-Myers type results);
- Gradient bounds for the heat semigroup;
- Spectral gap estimates for the sub-Laplacian.

Contents

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1 Introduction

Let $(\mathbb{M}, \theta, g)$ be a $2n+1$ smooth contact Riemannian manifold. On $\mathbb{M}$, there is a canonical diffusion operator L: The contact sub-Laplacian. This operator is not elliptic but only subelliptic in the sense of Fefferman-Phong [11] (see also [14] for a survey on subelliptic diffusion operators).
This lack of ellipticity makes the study of the geometrically relevant functional inequalities associated to L particularly delicate. Some methods have been developed in the literature but are local in nature (see [10], [13], [19]) and no global methods were known before the work by Baudoin-Garofalo [2], except in the three dimensional case (see [1], [17]). One of the main obstacles is the complexity of the theory of Jacobi vector fields (see [16]).

In the work [2], instead of dealing with Jacobi fields, the authors use the Bochner's method and proved that, if $\mathbb{M}$ is Sasakian, then under some geometric conditions (a lower bound on the Ricci curvature tensor of the Tanaka-Webster connection), the operator L satisfies a generalized curvature dimension inequality that we now describe. On $\mathbb{M}$, there is a canonical vector field, the Reeb vector field Z of the contact form θ, it is transverse to the kernel of θ.
Given the sub-Laplacian L and the first-order bilinear forms

$$\Gamma(f) = \frac{1}{2}\left(L(f^2) - 2fLf\right),$$

and

$$\Gamma^Z(f) = (Zf)^2,$$

we can introduce the following second-order differential forms:

$$\Gamma_2(f,g) = \frac{1}{2}\big[L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\big], \tag{1.1}$$

$$\Gamma_2^Z(f,g) = \frac{1}{2}\big[L\Gamma^Z(f,g) - \Gamma^Z(f,Lg) - \Gamma^Z(g,Lf)\big]. \tag{1.2}$$

The following basic result connecting the geometry of the contact manifold $\mathbb{M}$ to the analysis of its sub-Laplacian was then proved in [2]. It requires the contact structure on $\mathbb{M}$ to be of Sasakian type: A class of contact manifolds that contain very interesting examples (see [9], [26]) but that is somehow restrictive as we explain below.

Theorem 1.1 *Let $(\mathbb{M}, \theta)$ be a complete Sasakian contact manifold with dimension $2n+1$. The Tanaka-Webster Ricci tensor satisfies the bound*

$$\mathbf{Ric}_x(v,v) \ \geq \rho_1|v|^2, x \in \mathbb{M}, v \in \mathbf{Ker}(\theta),$$

if and only if for every smooth and compactly supported function f,

$$\Gamma_2(f) + 2\sqrt{\Gamma(f)\Gamma_2^Z(f)} \geq \frac{1}{2n}(Lf)^2 + \rho_1\Gamma(f) + \frac{n}{2}\Gamma^Z(f). \tag{1.3}$$

Observe that by linearization, the inequality (1.3) is equivalent to the fact that for every $\nu > 0$,

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{2n}(Lf)^2 + \left(\rho_1 - \frac{1}{\nu}\right)\Gamma(f) + \frac{n}{2}\Gamma^Z(f). \tag{1.4}$$

Theorem 1.1 opened the door to the study of global functional inequalities on contact Sasakian manifolds, like the log-Sobolev inequalities (see [6]), the Sobolev and isoperimetric inequalities (see [8]), the Li-Yau type gradient bounds for the heat kernel (see [2]) and the Gaussian upper and lower bounds for the heat kernel (see [7]). These inequalities were obtained through a systematic use of the heat semigroup associated to L and Bakry-Émery type computations [3], [4].

Our goal in this paper is to remove the assumption that $\mathbb{M}$ is a Sasakian manifold. The Sasakian condition is equivalent to the fact that the contact manifold carries a CR structure and that the Reeb vector field acts isometrically on the kernel of θ. This condition implies that the forms Γ and Γ^Z are intertwined in the sense that

$$\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)). \tag{1.5}$$

This condition is restrictive and many interesting examples of contact manifolds are not Sasakian. It is thus interesting to see if the Sasakian condition can be dropped. Our main result in that direction is the following theorem that shows the structure of the curvature dimension condition in the most general class of contact manifolds:

Theorem 1.2 *(See Theorem 3.6) Let $(\mathbb{M}, \theta, g)$ be a $2n+1$-dimensional contact Riemannian manifold. If some geometric conditions are satisfied, then there exist constants ρ_1, ρ_2 and ρ_3 such that for every $\nu > 0$ and smooth and compactly supported function f:*

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{2n}(Lf)^2 + \left(\rho_1 - \frac{1}{\nu}\right)\Gamma(f) + (\rho_2 - \rho_3\nu^2)\Gamma^Z(f). \tag{1.6}$$

The main difference with the Sasakian curvature dimension condition (1.4) is therefore the appearance of the strongly nonlinear term $-\rho_3\nu^2\Gamma^Z(f)$. It is noticeable that this new curvature dimension inequality appears as a special case of a general class of inequalities that was recently proposed in an abstract setting by F.Y. Wang in [25]. Our approach here is more of geometric nature, in the sense that our goal is to precisely understand what are the geometric bounds that imply a curvature dimension condition. As a consequence we get a very explicit curvature dimension condition.
As we will see, the new term makes the curvature dimension condition much more difficult to exploit. However, we can still address the following questions by using our new curvature dimension inequality:

1. **Bonnet-Myers type results** (See Theorem 4.2). We provide geometric conditions ensuring the compactness of $\mathbb{M}$;

2. **Stochastic completeness of the heat semigroup associated to the contact sub-Laplacian** (See Proposition 4.4). We prove that if the curvature dimension inequality (1.6) and an additional condition are satisfied, then the semigroup is stochastically complete.

3. **Poincaré inequality** (See Theorem 5.5). By using the generalized curvature dimension inequality to prove gradient bounds for the heat semigroup, we show that if (1.6) is satisfied with $\rho_1 - \frac{\kappa\sqrt{\rho_3}}{\sqrt{\rho_2}} > 0$, then for every smooth and compactly supported function f on $\mathbb{M}$:

$$\int_{\mathbb{M}} f^2 d\mu - \left(\int_{\mathbb{M}} f d\mu\right)^2 \leq \frac{\rho_2 + \kappa}{\rho_1\rho_2 - \kappa\sqrt{\rho_2\rho_3}} \int_{\mathbb{M}} \Gamma(f) d\mu.$$

As a consequence, $-L$ has a spectral gap of size bigger than $\frac{\rho_1\rho_2-\kappa\sqrt{\rho_2\rho_3}}{\rho_2+\kappa}$.

The paper is organized as follows. In Section 2, we introduce the geometric prerequisites that are needed in this work. The Section 3 is devoted to a careful analysis of the Bochner's type formulas that are needed to establish the generalized curvature dimension condition 1.6. These Bochner's formulas are related to some formulas obtained in a more general setting by Hladky in [12] but for our purpose, we need to do our computations in a very different way in order to extract the geometric quantities that are relevant for the Γ_2 calculus. In Section 4, we apply the generalized curvature dimension inequality to the study of the stochastic completeness of the subelliptic heat semigroup and to the problem of the compactness of the manifold. At the same occasion, we prove an estimate on the volume of geodesic balls. In Section 5, we study gradient bounds for the heat semigroup with purpose of proving the spectral gap inequality when $\rho_1 - \frac{\kappa\sqrt{\rho_3}}{\sqrt{\rho_2}} > 0$. In the last Section 6, we show how, by using stochastic analysis, it is possible to check some assumptions that are needed in the Section 5.

2 The sub-Laplacian of a contact Riemannian manifold

Let $(\mathbb{M}, \theta)$ be a $2n+1$-dimensional smooth contact manifold. On $\mathbb{M}$ there is a unique smooth vector field Z, the Reeb vector field, that satisfies

$$\theta(Z) = 1, \quad \mathcal{L}_Z(\theta) = 0,$$

where $\mathcal{L}_Z$ denotes the Lie derivative with respect to Z. The kernel of θ defines a $2n$ dimensional subbundle of $\mathbb{M}$ which shall be referred to as the set of horizontal directions and denoted $\mathcal{H}(M)$. The vector field Z is transverse to $\mathcal{H}(M)$ and will be referred to as the vertical direction.
According to [24], it is always possible to find a Riemannian metric g and a $(1,1)$-tensor field J on $\mathbb{M}$ so that for every vector fields X, Y

$$g(X, Z) = \theta(X), \quad J^2 = -I + \theta \otimes Z, \quad g(X, JY) = (d\theta)(X, Y).$$

The triple $(\mathbb{M}, \theta, g)$ is called a contact Riemannian manifold, a geometric structure well studied by Tanno in [24]. On a contact Riemannian manifold, the Riemannian structure of $\mathbb{M}$ is actually often confined to the background whereas the sub-Riemannian structure of $\mathbb{M}$ carries more fundamental informations about the contact structure (see [5], [17] [24]). If $f : \mathbb{M} \to \mathbb{R}$ is a smooth function, we denote by $\nabla_{\mathcal{H}} f$ the horizontal gradient of f which is defined as the projection of the Riemannian gradient of f onto the horizontal space $\mathcal{H}(\mathbb{M})$. The sub-Laplacian L of the contact Riemannian manifold $(\mathbb{M}, \theta, g)$ is then defined as the generator of the symmetric Dirichlet form

$$\mathcal{E}(f, g) = \int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g \rangle d\mu,$$

where μ is the Borel measure given the $2n + 1$ volume form $\theta \wedge (d\theta)^n$. The diffusion operator L is not elliptic but subelliptic of order $1/2$ (see [5]). We can observe that, as a direct consequence of the definition of L, we have

$$L = \Delta - Z^2,$$

where Δ is the Laplace-Beltrami of the Riemannian structure $(\mathbb{M}, g)$. The following lemma will be useful:

Lemma 2.1 *If the Riemannian manifold $(\mathbb{M}, g)$ is complete, then L is essentially self-adjoint on the space $C_0^\infty(\mathbb{M})$ of smooth and compactly supported functions..*

Proof. If $(\mathbb{M}, g)$ is complete, then from [20], there exists a sequence h_n in $C_0^\infty(\mathbb{M})$ such that $\|\nabla_{\mathcal{H}} f\|_\infty + \|Zf\|_\infty \to 0$ when $n \to \infty$. In particular $\|\nabla_{\mathcal{H}} f\|_\infty \to 0$, and thus from [21], L is essentially self-adjoint on the space $C_0^\infty(\mathbb{M})$. $\square$

In the sequel of the paper we always assume that $(\mathbb{M}, g)$ is complete.

We denote by ∇^R the Levi-Civita connection on $\mathbb{M}$. The following $(1, 2)$ tensor field Q on $(\mathbb{M}, g)$ that was introduced by Tanno in [24] as follows:

$$Q(X, Y) = (\nabla_X^R J)Y + [(\nabla_Y^R \theta)JX]Z + \theta(X)J(\nabla_Y^R Z)$$

will play a pervasive role in this paper. A fundamental result due to Tanno is that $(\mathbb{M}, \theta, J|_{\mathcal{H}(\mathbb{M})})$ is a strongly pseudo convex CR manifold if and only if $Q = 0$
Besides the Riemannian connection ∇^R, there is a canonical sub-Riemannian connection that was introduced by Tanno in [24] and which generalizes the Tanaka-Webster connection of the CR manifolds. This connection denoted by ∇ in the sequel, is much more naturally associated with the study of the sub-Laplacian L. In terms of the Riemannian connection, the Tanno's connection writes for every vector fields X, Y,

$$\nabla_X Y = \nabla_X^R Y + \theta(X)JY - \theta(Y)\nabla_X^R Z + [(\nabla_X^R \theta)Y]Z.$$

This connection ∇ is more intrinsically characterized as follows:

Proposition 2.2 (S. Tanno, [24]) *The connection ∇ on $(\mathbb{M}, \theta, g)$ is the unique linear connection that satisfies:*

1. $\nabla\theta = 0$;
2. $\nabla Z = 0$;
3. $\nabla g = 0$;
4. $T(X, Y) = d\theta(X, Y)Z$ *for any* $X, Y \in \mathcal{H}(\mathbb{M})$;
5. $T(Z, JX) = -JT(Z, X)$ *for any vector field* X;
6. $(\nabla_X J)Y = Q(Y, X)$ *for any vector fields* X, Y.

where $T(\cdot, \cdot)$ is the torsion tensor with respect to ∇.

If X is a horizontal vector field, so is $T(Z, X)$. As a consequence if we define $\tau(X) = T(Z, X)$, τ is a symmetric horizontal endomorphism which satisfies $\tau \circ J + J \circ \tau = 0$. In the context of CR manifolds, τ is referred to as the pseudo-Hermitian torsion. We can observe that $\tau = 0$ is equivalent to the fact that the contact structure is of K type (see [24]).

For our purpose, it will be expedient to work in local frames that are adapted to the contact structure. If $X_1, X_2, \cdots, X_{2n}$ is a local orthonormal frame of $\mathcal{H}(\mathbb{M})$, all the local geometry of the contact manifold is contained into the structure coefficients that are defined by

$$[X_i, X_j] = \sum_{k=1}^{2n} w_{ij}^k X_k + \gamma_{ij} Z, \quad [X_i, Z] = \sum_{j=1}^{2n} \delta_i^j X_j \tag{2.7}$$

where w_{ij}^k, γ_{ij}, δ_i^j are smooth functions. It is easy to see that

$$w_{ij}^k = -w_{ji}^k, \quad \gamma_{ij} = -\gamma_{ji}, \quad i, j, k = 1, \cdots 2n. \tag{2.8}$$

In the local frame $\{X_1, \cdots, X_{2n}, Z\}$ as above, the sub-Laplacian L can be written

$$L = -\sum_{i=1}^{2n} X_i^* X_i,$$

where X_i^* is the formal adjoint of X_i with respect to the volume measure μ. From (2.7), we obtain

$$X_i^* = -X_i + \sum_{k=1}^{2n} w_{ik}^k.$$

Hence, we can write locally

$$L = \sum_{i=1}^{2n} X_i^2 + X_0,$$

where

$$X_0 = -\sum_{i,k=1}^{2n} w_{ik}^k X_i. \tag{2.9}$$

By (2.7), one can then easily calculate the Christoffel's symbols of the sub-Riemannian connection:

$$\nabla_{X_i} X_j = \sum_{k=1}^{2n} \Gamma_{ij}^k X_k, \quad \nabla_Z X_i = \frac{1}{2}\sum_{k=1}^{2n}(\delta_k^i - \delta_i^k) X_k$$

where $\Gamma_{ij}^k = \frac{1}{2}(w_{ij}^k + w_{ki}^j + w_{kj}^i)$. It is also easy to see that

$$\tau(X_i) = \frac{1}{2}\sum_{k=1}^{2n}(\delta_k^i + \delta_i^k) X_k, \quad T(X_j, X_k) = -\gamma_{jk} Z \quad \text{and} \quad JX_i = \sum_{j=1}^{2n} \gamma_{ij} X_j.$$

In the case of CR Sasakian manifolds, in addition to the relations in (2.8), we also have the skew-symmetry of the δ_i^js, i.e., $\delta_i^j = -\delta_j^i$ for all $i, j = 1, \cdots, 2n$, which implies that the torsion τ vanishes (see [2]).
In our general case, though the skew-symmetry is no more satisfied, we can still always find a basis such that the diagonal entries of τ vanish. i.e., $\delta_i^i = 0$, for all $i = 1, \cdots, 2n$. Indeed, let λ be an eigenvalue of τ and X a corresponding eigenvector. Since $\tau \circ J + J \circ \tau = 0$, this implies that $-\lambda$ is also an eigenvalue of τ. Hence τ is similar to the diagonal matrix

$$A = \begin{pmatrix} A_1 & & \\ & \cdots & \\ & & A_n \end{pmatrix}, \quad A_i = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}, \quad i = 1, \cdots, n.$$

Since we have $A_i \sim \begin{pmatrix} 0 & \lambda_i \\ \lambda_i & 0 \end{pmatrix} := \tilde{A}_i$, thus $A \sim \begin{pmatrix} \tilde{A}_1 & & \\ & \cdots & \\ & & \tilde{A}_n \end{pmatrix}$.
In the sequel, we thus always choose the local frame such that $\delta_i^i = 0$, $i = 1, \cdots, 2n$.

3 The generalized curvature dimension inequality

3.1 Bochner's formulas

Our first goal will be to work out the Bochner's type formulas for the sub-Laplacian L. We follow the methods of [2] and use the Γ_2 formalism introduced in [4].
Let us consider the first order differential bilinear form:

$$\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf), \quad f, g \in C^\infty(\mathbb{M}),$$

and observe that

$$\Gamma(f, g) = \langle \nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g \rangle,$$

where $\nabla_{\mathcal{H}}$ is the horizontal gradient. $\Gamma(f) = \Gamma(f,f)$ is known as *le carré du champ*. Similarly we define for every $f, g \in C^\infty(\mathbb{M})$,

$$\Gamma^Z(f,g) = \langle \nabla_{\mathcal{V}} f, \nabla_{\mathcal{V}} g \rangle,$$

where $\nabla_{\mathcal{V}}$ is the vertical gradient of $\mathbb{M}$. We also introduce the second order differential bilinear forms:

$$\Gamma_2(f,g) = \frac{1}{2}(L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)) \tag{3.10}$$

and

$$\Gamma_2^Z(f,g) = \frac{1}{2}(L\Gamma^Z(f,g) - \Gamma^Z(f,Lg) - \Gamma^Z(g,Lf)). \tag{3.11}$$

Throughout the Section, we work in a local frame that satisfies

$$[X_i, X_j] = \sum_{k=1}^{2n} w_{ij}^k X_k + \gamma_{ij} Z, \quad [X_i, Z] = \sum_{j=1}^{2n} \delta_i^j X_j$$

with $\delta_i^i = 0$.
The following tensorial quantity will play a crucial role in our discussion.

Definition 3.1 *Let* $\mathbf{Ric}(\cdot,\cdot)$ *and* $T(\cdot,\cdot)$ *respectively denote the Ricci and torsion tensors of the sub-Riemannian connection* ∇. *For* $f \in C^\infty(\mathbb{M})$ *we define:*

$$\begin{aligned} &\mathcal{R}(f,f) \\ =&\mathbf{Ric}(\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} f) + \frac{n}{2}\|\nabla_{\mathcal{V}} f\|^2 - \sum_{l,k=1}^{2n} \left(\left((\nabla_{X_l} T)(X_l, X_k) f (X_k f) \right) + T(X_l, T(X_l, X_k)) f X_k f \right). \end{aligned} \tag{3.12}$$

From its definition, it is obvious that $\mathcal{R}$ is an intrinsic first order differential bilinear form on $\mathbb{M}$. The following proposition provides its computations in terms of the structure constants of the local frame.

Lemma 3.2 *We have:*

$$\mathcal{R}(f,f) = \sum_{k,l=1}^{2n} \mathcal{R}_{kl} X_k f X_l f + \sum_{k=1}^{2n} \left(\sum_{l,j=1}^{2n} w_{jl}^l \gamma_{kj} + \sum_{1 \le l < j \le 2n} w_{lj}^k \gamma_{lj} - \sum_{j=1}^{2n} X_j \gamma_{kj} \right) Z f X_k f + \frac{n}{2}(Zf)^2,$$

with

$$\mathcal{R}_{kl} = \sum_{j=1}^{2n} \gamma_{kj} \delta_j^l + \sum_{j=1}^{2n} (X_l w_{kj}^j - X_j w_{lj}^k) + \sum_{i,j=1}^{2n} w_{ji}^i w_{kj}^l - \sum_{j=1}^{2n} w_{ki}^i w_{li}^i + \frac{1}{2} \sum_{1 \le i < j \le 2n} \left(w_{ij}^l w_{ij}^k - (w_{lj}^i + w_{li}^j)(w_{kj}^i + w_{ki}^j) \right).$$

Proof. We write $\mathcal{R}(f,f)$ as follows

$$\mathcal{R}(f,f)=\mathcal{R}_I(f,f)+\mathcal{R}_{II}(f,f)+\mathcal{R}_{III}(f,f),$$

where

$$\begin{aligned}
\mathcal{R}_I(f,f) &= \sum_{l,k=1}^{2n}\Big(\mathbf{Ric}(X_l,X_k)X_lfX_kf+T(X_l,T(X_l,X_k))fX_kf\Big), &(3.13)\\
\mathcal{R}_{II}(f,f) &= -\sum_{k,l=1}^{2n}\Big((\nabla_{X_l}T)(X_l,X_k)f(X_kf)\Big), &(3.14)\\
\mathcal{R}_{III}(f,f) &= \frac{n}{2}(Zf)^2. &(3.15)
\end{aligned}$$

Straightforward but tedious calculations show that

$$\begin{aligned}
\sum_{l,k=1}^{2n}T(X_l,T(X_l,X_k))fX_kf &= -\sum_{l,k,j=1}^{2n}\frac{\delta_k^j+\delta_j^k}{2}\gamma_{jl}X_lfX_kf \qquad (3.16)\\
\sum_{l,k=1}^{2n}\mathbf{Ric}(X_l,X_k)X_lfX_kf &= \sum_{i,j,l,k=1}^{2n}\Big(\Gamma_{lk}^i\Gamma_{ji}^j-\Gamma_{jk}^i\Gamma_{li}^j-w_{jl}^i\Gamma_{ik}^j\Big)X_lfX_kf\\
&+\sum_{j,l,k=1}^{2n}\Big((X_j\Gamma_{lk}^j)-(X_l\Gamma_{jk}^j)\Big)X_lfX_kf-\sum_{l,k,j=1}^{2n}\gamma_{jl}\frac{\delta_j^k-\delta_k^j}{2}X_lfX_kf,
\end{aligned}$$

which implies that

$$\mathcal{R}_I(f,f) = \sum_{k,l=1}^{2n}\mathcal{R}_{kl}X_kfX_lf.$$

We also calculate in a direct way that

$$\mathcal{R}_{II}(f,f) = \sum_{k=1}^{2n}\left(\sum_{l,j=1}^{2n}w_{jl}^l\gamma_{kj}+\sum_{1\le l<j\le 2n}w_{lj}^k\gamma_{lj}-\sum_{j=1}^{2n}X_j\gamma_{kj}\right)ZfX_kf.$$

By combining the above terms we have the lemma. □

With these preliminary results in hands, we can now turn to the proof of the horizontal Bochner's formula:

Theorem 3.3 *For every $f\in C^\infty(\mathbb{M})$, the following* **Horizontal Bochner formula** *holds:*

$$\Gamma_2(f)=\|\nabla_{\mathcal{H}}^2f\|^2+\mathcal{R}(f,f)-2\sum_{i,j=1}^{2n}\gamma_{ij}(X_jZf)(X_if). \qquad (3.17)$$

Proof. It is enough to prove (3.17) in the local frame $\{X_1, ..., X_{2n}, Z\}$. Observe that

$$X_i X_j f = f_{,ij} + \frac{1}{2}[X_i, X_j]f,$$

where we have let

$$f_{,ij} = \frac{1}{2}(X_i X_j + X_j X_i)f. \tag{3.18}$$

Using (2.7), we find

$$X_i X_j f = f_{,ij} + \frac{1}{2}\sum_{\ell=1}^{2n} \omega_{ij}^{\ell} X_\ell f + \frac{1}{2}\gamma_{ij} Z f. \tag{3.19}$$

Now, starting from the definition (3.10) of $\Gamma_2(f)$, we obtain

$$\Gamma_2(f) = \sum_{i=1}^{2n} X_i f [X_0, X_i] f - 2\sum_{i,j=1}^{2n} X_i f [X_i, X_j] X_j f + \sum_{i,j=1}^{2n} X_i f [[X_i, X_j], X_j] f + \sum_{i,j=1}^{2n} (X_j X_i f)^2,$$

where X_0 is defined by (2.9). From (3.19) we have

$$\begin{aligned}\sum_{i,j=1}^{2n} (X_j X_i f)^2 = \sum_{i,j=1}^{2n} f_{,ij}^2 + \frac{1}{2}\sum_{1\le i<j\le 2n}\left(\sum_{\ell=1}^{2n}\omega_{ij}^{\ell} X_\ell f\right)^2 + \frac{1}{2}\sum_{1\le i<j\le 2n}(\gamma_{ij} Z f)^2 \\ + \sum_{1\le i<j\le 2n}\sum_{\ell=1}^{2n}\omega_{ij}^{\ell}\gamma_{ij} Z f X_\ell f,\end{aligned}$$

and therefore,

$$\begin{aligned}\Gamma_2(f) = & \sum_{i,j=1}^{2n} f_{,ij}^2 - 2\sum_{i,j=1}^{2n} X_i f [X_i, X_j] X_j f + \sum_{i,j=1}^{2n} X_i f [[X_i, X_j], X_j] f \\ & + \sum_{i=1}^{2n} X_i f [X_0, X_i] f + \frac{1}{2}\sum_{1\le i<j\le 2n}\left(\sum_{\ell=1}^{2n}\omega_{ij}^{\ell} X_\ell f\right)^2 + \frac{1}{2}\sum_{1\le i<j\le 2n}(\gamma_{ij} Z f)^2 \\ & + \sum_{1\le i<j\le 2n}\sum_{\ell=1}^{2n}\omega_{ij}^{\ell}\gamma_{ij} Z f X_\ell f.\end{aligned} \tag{3.20}$$

By plugging in (2.7) and completing the square, we obtain

$$\begin{aligned}\Gamma_2(f) = & \sum_{\ell=1}^{2n}\left(f_{,\ell\ell} - \sum_{i=1}^{2n}\omega_{i\ell}^{\ell} X_i f\right)^2 + 2\sum_{1\le \ell<j\le 2n}\left(f_{,j\ell} - \sum_{i=1}^{2n}\frac{\omega_{ij}^{\ell} + \omega_{i\ell}^{j}}{2} X_i f\right)^2 \\ & - 2\sum_{i,j=1}^{2n}\gamma_{ij} X_j Z f \; X_i f + \mathcal{R}(f),\end{aligned}$$

where we used the fact that $\sum_{1\le i<j\le 2n}(\gamma_{ij}Zf)^2 = n(Zf)^2$. At last, we complete the proof of (3.17) by realizing that the square of the Hilbert-Schmidt norm of the horizontal Hessian $\nabla^2_{\mathcal{H}}f$ is given by

$$\|\nabla^2_{\mathcal{H}}f\|^2 = \sum_{\ell=1}^{2n}\left(f_{,\ell\ell} - \sum_{i=1}^{2n}\omega_{i\ell}^{\ell}X_if\right)^2 + 2\sum_{1\le \ell<j\le 2n}\left(f_{,j\ell} - \sum_{i=1}^{2n}\frac{\omega_{ij}^{\ell}+\omega_{i\ell}^{j}}{2}X_if\right)^2. \tag{3.21}$$

$\square$

Our next goal is to derive a vertical Bochner's formula. We first give the formula in terms of the structure constants and will provide the tensorial expressions afterwards.

Theorem 3.4 *For every $f \in C^\infty(\mathbb{M})$,*

$$\Gamma_2^Z(f) = \sum_{i=1}^{2n}(X_iZf)^2 + \frac{1}{2}\sum_{i,l=1}^{2n}(\delta_i^l+\delta_l^i)(X_iX_lf+X_lX_if)Zf + \sum_{i,l=1}^{2n}\left(X_i\delta_i^l - \sum_{k=1}^{2n}w_{ik}^k\delta_i^l + \sum_{k=1}^{2n}Zw_{lk}^k\right)X_lfZf. \tag{3.22}$$

Proof. From (3.11), we know that

$$\Gamma_2^Z(f) = \Gamma(Zf) + [L,Z]fZf. \tag{3.23}$$

Moreover, since

$$[L,Z]f \quad = \quad [X_0,Z]f + \sum_{i=1}^{2n}(X_i[X_i,Z]f + [X_i,Z]X_if)$$

we can easily compute that

$$[L,Z]f = -\sum_{i,k,l=1}^{2n}w_{ik}^k\delta_i^lX_lf + \sum_{l,k=1}^{2n}(Zw_{lk}^k)X_lf + \sum_{i,l=1}^{2n}(X_i\delta_i^l)X_lf + \frac{1}{2}\sum_{i,l=1}^{2n}(\delta_i^l+\delta_l^i)(X_iX_l+X_lX_i)f. \tag{3.24}$$

Plug this expression back in (3.23), we have the expression for $\Gamma_2^Z(f)$. $\square$

To stress that the formula, of course does not depend on the local frame, we can rewrite it as follows:

Theorem 3.5 *For any smooth function $f \in C^\infty(\mathbb{M})$, we have*

$$\begin{aligned}\Gamma_2^Z(f) =& \|\nabla_{\mathcal{H}}\nabla_{\mathcal{V}}f\|^2 + \mathbf{Ric}(\nabla_{\mathcal{H}}f,\nabla_{\mathcal{V}}f)\\ &+ 2\sum_{i,l=1}^{2n}\tau(X_i)X_ifZf + \sum_{k=1}^{2n}(\nabla_{X_k}T)(Z,X_k)fZf - 2\sum_{k=1}^{2n}\nabla_{\tau(X_k)}X_kfZf\end{aligned}$$

Proof. Since

$$(\nabla_{X_k} T)(Z, X_i) \quad = \quad \nabla_{X_k}(\tau(X_i)) - \tau(\nabla_{X_k} X_i),$$

we have that

$$(\nabla_{X_k} T)(Z, X_k) \quad = \quad \frac{1}{2}\sum_{l=1}^{2n} X_k(\delta_k^l + \delta_l^k) X_l + \frac{1}{2}\sum_{l,j=1}^{2n} (\delta_k^l + \delta_l^k)\Gamma_{kl}^j X_j - \frac{1}{2}\sum_{l,j=1}^{2n} w_{lk}^k \left(\delta_l^j + \delta_j^l\right) X_j.$$

and simple calculations give us

$$\begin{aligned}\mathbf{Ric}(Z, X_i) =& \frac{1}{2}\sum_{j=1}^{2n} X_j(\delta_j^i - \delta_i^j) + \frac{1}{2}\sum_{j,k=1}^{2n} w_{jk}^j(\delta_k^i - \delta_i^k) \\ & - \sum_{j=1}^{2n} Z w_{ji}^j - \frac{1}{2}\sum_{k,j=1}^{2n} \Gamma_{ji}^k(\delta_j^k - \delta_k^j) - \sum_{k,j=1}^{2n} \delta_j^k \Gamma_{ki}^j.\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned} & \sum_{i=1}^{2n} \mathbf{Ric}(Z, X_i) X_i f Z f + \sum_{k=1}^{2n} (\nabla_{X_k} T)(Z, X_k) f Z f \\ = \quad & \sum_{i,j=1}^{2n} X_j \delta_j^i X_i f Z f + \sum_{i,j,k=1}^{2n} w_{jk}^j \delta_k^i X_i f Z f - \sum_{i,j=1}^{2n} (Z w_{ji}^j) X_i f Z f \\ + \quad & \left(\frac{1}{2}\sum_{i,j,k=1}^{2n} (\delta_k^j + \delta_j^k)\Gamma_{kj}^i X_i f - \frac{1}{2}\sum_{i,j,k=1}^{2n} \Gamma_{ji}^k(\delta_j^k - \delta_k^j) X_i f - \sum_{i,j,k=1}^{2n} \delta_j^k \Gamma_{ki}^j X_i f\right) Z f.\end{aligned}$$

By taking into account

$$\Gamma_{ji}^k = \Gamma_{kj}^i - (w_{ij}^k + w_{ik}^j) = w_{kj}^i - \Gamma_{kj}^i, \qquad \Gamma_{ki}^j = -\Gamma_{kj}^i,$$

we have that

$$\frac{1}{2}\sum_{i,j,k=1}^{2n} (\delta_k^j + \delta_j^k)\Gamma_{kj}^i X_i f - \frac{1}{2}\sum_{i,j,k=1}^{2n} \Gamma_{ji}^k(\delta_j^k - \delta_k^j) X_i f - \sum_{i,j,k=1}^{2n} \delta_j^k \Gamma_{ki}^j X_i f = \frac{1}{2}\sum_{i,j,k=1}^{2n} (w_{ij}^k + w_{ik}^j)(\delta_k^j + \delta_j^k) X_i f.$$

Moreover, notice that

$$\frac{1}{2}\sum_{i,j,k=1}^{2n} \left(\delta_k^j + \delta_j^k\right)\left(w_{ij}^k + w_{ik}^j\right) X_i f Z f = 2\sum_{k=1}^{2n} \nabla_{\tau(X_k)} X_k f Z f, \tag{3.25}$$

so that we can write

$$\sum_{i,j=1}^{2n} X_j \delta_j^i X_i f Zf + \sum_{i,j,k=1}^{2n} w_{jk}^j \delta_k^i X_i f Zf - \sum_{i,j=1}^{2n} (Z w_{ji}^j) X_i f Zf$$

$$= \sum_{i=1}^{2n} \mathbf{Ric}(Z, X_i) X_i f Zf + \sum_{k=1}^{2n} (\nabla_{X_k} T)(Z, X_k) f Zf - 2 \sum_{k=1}^{2n} \nabla_{\tau(X_k)} X_k f Zf. \quad (3.26)$$

At the end we conclude the proposition by comparing with the expression in (3.22). $\square$

3.2 Generalized curvature dimension bounds

With the two Bochner's formulas in hands, we are now ready to give the suitable curvature dimension conditions on contact manifolds. To this purpose, we introduce the relevant geometric quantities. As in the previous subsection, we work in a local frame.

The vector field

$$V = \sum_{i=1}^{2n} \mathbf{Ric}(Z, X_i) X_i + (\nabla_{X_i} T)(Z, X_i),$$

obviously does not depend on the choice of the local frame and is therefore an intrinsic invariant of the manifold. In terms of the structure constants, we compute

$$V = \sum_{i,j,l=1}^{2n} \left(\frac{\delta_j^l + \delta_l^j}{2} \right) \left(w_{il}^j + w_{ij}^l \right) X_i + \sum_{i=1}^{2n} \left(\sum_{j=1}^{2n} X_j \delta_j^i - \sum_{j,k=1}^{2n} w_{jk}^k \delta_j^i + \sum_{k=1}^{2n} Z w_{ik}^k \right) X_i.$$

We then consider the first-order quadratic differential form defined for $f \in C^\infty(\mathbb{M})$ by

$$\tau_2(f) = \sum_{l,k=1}^{2n} T(X_l, T(X_l, X_k)) f X_k f,$$

and the horizontal trace of the Tanno tensor Q which is the vector field given by $\mathbf{Tr}_\mathcal{H} Q := \sum_{l=1}^{2n} Q(X_l, X_l) = \sum_{l=1}^{2n} (\nabla_{X_l} J) X_l$. Our main result is the following:

Theorem 3.6 *Assume there exist constants $c_1 \in \mathbb{R}$, $c_2 \geq 0$, $c_3 \geq 0$ and $\iota \geq 0$ such that for every $f \in C^\infty(\mathbb{M})$,*

$$\mathbf{Ric}(\nabla_\mathcal{H} f) + \tau_2(f) \geq c_1 \|\nabla_\mathcal{H} f\|^2, \|(\mathbf{Tr}_\mathcal{H} Q) f\|^2 \leq c_2 \|\nabla_\mathcal{H} f\|^2, \quad (3.27)$$

$$\|V f\|^2 \leq c_3 \|\nabla_\mathcal{H} f\|^2, \quad \|\tau(\nabla_\mathcal{H} f)\|^2 \leq \iota \|\nabla_\mathcal{H} f\|^2.$$

Then for all $\nu > 0$ and $f \in C^\infty(\mathbb{M})$,

$$\Gamma_2(f) + \nu \Gamma_2^Z(f) \geq \frac{1}{2n} (Lf)^2 + \left(c_1 - \frac{1}{\nu} \right) \Gamma(f) - (c_2 + c_3 \nu) \sqrt{\Gamma(f) \Gamma^Z(f)} + \left(\frac{n}{2} - \frac{\iota}{4} \nu^2 \right) \Gamma^Z(f).$$

Proof. To derive the generalized curvature-dimension inequality, let us first introduce the first-order differential forms $\mathcal{U}$ and $\mathcal{T}$ in the local frame $\{X_1, \cdots, X_{2n}\}$ such that

$$\mathcal{T}(f,f) = \sum_{k=1}^{2n} \|T(X_k, \nabla_{\mathcal{H}} f)^2\|, \quad \mathcal{U}(f,f) = \sum_{k=1}^{2n} \|\tau(X_k)\|^2 (Zf)^2. \tag{3.28}$$

A simple computation shows that

$$\mathcal{U}(f,f) = \sum_{j,l=1}^{2n} \left(\frac{\delta_j^l + \delta_l^j}{2} \right)^2 (Zf)^2, \quad \mathcal{T}(f,f) = \sum_{j=1}^{2n} \left(\sum_{i=1}^{2n} \gamma_{ij} X_i f \right)^2. \tag{3.29}$$

Let us also consider $\mathcal{S}(f) = VfZf$ so that

$$\mathcal{S}(f) = \mathbf{Ric}(\nabla_{\mathcal{V}} f, \nabla_{\mathcal{H}} f) + \sum_{i=1}^{2n} (\nabla_{X_i} T)(Z, X_i) f Z f. \tag{3.30}$$

From (3.17) and (3.22), by using the fact that $\delta_i^i = 0$, we have that

$$\begin{aligned}
\Gamma_2(f,f) + \nu \Gamma_2^Z(f,f) &= \sum_{l=1}^{2n} \left(X_l^2 f - \sum_{i=1}^{2n} w_{il}^l X_i f \right)^2 - 2 \sum_{i,j=1}^{2n} \gamma_{ij} (X_j Z f)(X_i f) \\
&+ \nu \sum_{i=1}^{2n} (X_i Z f)^2 + 2\nu \sum_{1 \le l < j \le 2n} \left(\frac{\delta_j^l + \delta_l^j}{2} \right) \left(\frac{X_j X_l + X_l X_j}{2} \right) f Z f \\
&+ 2 \sum_{1 \le l < j \le 2n} \left(\left(\frac{X_l X_j + X_j X_l}{2} \right) f - \sum_{i=1}^{2n} \left(\frac{w_{il}^j + w_{ij}^l}{2} \right) X_i f \right)^2 \\
&+ \nu \sum_{l=1}^{2n} \left(\sum_{i=1}^{2n} X_i \delta_i^l - \sum_{i,k=1}^{2n} w_{ik}^k \delta_i^l + \sum_{k=1}^{2n} Z w_{lk}^k \right) X_l f Z f + \mathcal{R}(f,f).
\end{aligned}$$

We write the above equation as follows:

$$\Gamma_2(f,f) + \nu \Gamma_2^Z(f,f) = \mathcal{B}_I + \mathcal{B}_{II} + \mathcal{B}_{III} + \nu \sum_{l=1}^{2n} \left(\sum_{i=1}^{2n} X_i \delta_i^l - \sum_{i,k=1}^{2n} w_{ik}^k \delta_i^l + \sum_{k=1}^{2n} Z w_{lk}^k \right) X_l f Z f + \mathcal{R}(f,f),$$

where

$$\mathcal{B}_I = \sum_{l=1}^{2n} \left(X_l^2 f - \sum_{i=1}^{2n} w_{il}^l X_i f \right)^2,$$

$$\mathcal{B}_{II} = -2 \sum_{i,j=1}^{2n} \gamma_{ij}(X_j Z f)(X_i f) + \nu \sum_{i=1}^{2n} (X_i Z f)^2,$$

$$\mathcal{B}_{III} = 2 \sum_{1\le l<j\le 2n} \left(\left(\frac{X_l X_j + X_j X_l}{2} \right) f - \sum_{i=1}^{2n} \left(\frac{w_{il}^j + w_{ij}^l}{2} \right) X_i f \right)^2$$
$$+2\nu \sum_{1\le l<j\le 2n} \left(\frac{\delta_j^l + \delta_l^j}{2} \right) \left(\frac{X_j X_l + X_l X_j}{2} \right) f Z f$$

Hence from Cauchy-Schwartz inequality we obtain

$$\mathcal{B}_I \ge \frac{1}{2n}(Lf)^2.$$

Also we can easily see that

$$\mathcal{B}_{II} \ge -\frac{1}{\nu} \sum_{j=1}^{2n} \left(\sum_{i=1}^{2n} \gamma_{ij} X_i f \right)^2,$$

and

$$\mathcal{B}_{III} \ge 2\nu \sum_{1\le l<j\le 2n} \sum_{i=1}^{2n} \left(\frac{\delta_j^l + \delta_l^j}{2} \right) \left(\frac{w_{il}^j + w_{ij}^l}{2} \right) X_i f Z f - \frac{\nu^2}{2} \sum_{1\le l<j\le 2n} \left(\left(\frac{\delta_j^l + \delta_l^j}{2} \right) Z f \right)^2.$$

Hence we have

$$\Gamma_2(f,f) + \nu \Gamma_2^Z(f,f) \ge \frac{1}{2n}(Lf)^2 - \frac{\nu^2}{4}\mathcal{U}(f) + \nu \mathcal{S}(f) + \mathcal{R}(f) - \frac{1}{\nu}\mathcal{T}(f),$$

and the conclusion easily follows from the fact that

$$\mathcal{T}(f) = \sum_{k=1}^{2n} \langle J\nabla_{\mathcal{H}} f, X_k \rangle^2 = \|J\nabla_{\mathcal{H}} f\|^2 = \Gamma(f).$$

□

In the case of Sasakian manifolds, we have $V = 0$, $Q = 0$, $\tau = 0$ and we recover the curvature dimension inequality introduced in [2].
In view of Theorem 3.6, it is then natural to set the following definition:

Definition 3.7 *We say that $\mathbb{M}$ satisfies the* ***generalized curvature-dimension inequality*** *$CD(\rho_1, \rho_2, \rho_3, \kappa, m)$ with respect to L and Γ^Z if there exist constants $\rho_1, \rho_2 \in \mathbb{R}$, $\rho_3 > 0$, $\kappa > 0$, $0 < m \leq \infty$ such that the inequality*

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{m}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f) + \left(\rho_2 - \rho_3\nu^2\right)\Gamma^Z(f)$$

hold for all $f \in C^\infty(\mathbb{M})$ and every $\nu > 0$.

In particular, under the assumptions of Theorem 3.6 we easily see that the curvature-dimension inequality $CD(\rho_1, \rho_2, \rho_3, 1, 2n)$ holds for every $z > 0$, $w > 0$, where $\rho_1 = c_1 - \frac{c_2 z}{2} - \frac{c_3 w}{2}$, $\rho_2 = \frac{n}{2} - \frac{c_2}{2z}$, $\rho_3 = \frac{c_3}{2w} + \frac{\iota}{4}$.
It is very interesting to observe that Theorem 3.6 admits a partial converse.

Theorem 3.8 *Assume that there exist constants c_1, c_2, c_3 and ι such that for every $\nu > 0$ and $f \in C^\infty(\mathbb{M})$,*

$$\Gamma_2(f)+\nu\Gamma_2^Z(f) \geq \frac{1}{2n}(Lf)^2+\left(c_1 - \frac{1}{\nu}\right)\Gamma(f)-(c_2 + c_3\nu)\sqrt{\Gamma(f)\Gamma^Z(f)}+\left(\frac{n}{2} - \frac{\iota}{4}\nu^2\right)\Gamma^Z(f).$$

then, we have for every $f \in C^\infty(\mathbb{M})$,

$$\mathbf{Ric}(\nabla_{\mathcal{H}} f) + \tau_2(f) \geq c_1\|\nabla_{\mathcal{H}} f\|^2$$

and

$$\|\tau(\nabla_{\mathcal{H}} f)\|^2 \leq \iota\|\nabla_{\mathcal{H}} f\|^2.$$

Proof. We first observe that under our assumptions the curvature-dimension inequality $CD(\rho_1, \rho_2, \rho_3, 1, 2n)$ holds for every $z > 0$, $w > 0$, where $\rho_1 = c_1 - \frac{c_2 z}{2} - \frac{c_3 w}{2}$, $\rho_2 = \frac{n}{2} - \frac{c_2}{2z}$, $\rho_3 = \frac{c_3}{2w} + \frac{\iota}{4}$.
For a fixed $x_0 \in \mathbb{M}$, $u \in \mathcal{H}_{x_0}(\mathbb{M})$, $v \in \mathcal{V}_{x_0}(\mathbb{M})$, let $\{X_1, X_2, \cdots, X_{2n}, Z\}$ be a local adapted frame around x_0. First we claim that for $\nu > 0$, we can find a function $f \in C^\infty(\mathbb{M})$ satisfying:

(i) $\nabla_{\mathcal{H}} f(x_0) = u$,

(ii) $\nabla_{\mathcal{V}} f(x_0) = Zf(x_0) = v$,

(iii) $\left(\nabla_{\mathcal{H}}^2 f(x_0)\right)_{l,j} = \frac{\nu}{2}\left(\frac{\delta_j^l + \delta_l^j}{2}\right)(x_0)v$,

(iv) $X_j Z f(x_0) = \frac{1}{\nu}\sum_{i=1}^{2n} \gamma_{ij}(x_0)u_i$, for all $j = 1, \cdots, 2n$.

To prove this, let (U, ϕ) be a local chart at x_0, such that $\phi(0) = x_0$ and in U we have $X_j = \frac{\partial}{\partial x_j}$, $j = 1, ..., 2n$, $Z = \frac{\partial}{\partial z}$. Then the existence of f follows immediately by the existence of functions $f_1 \in C^\infty(\mathbb{M})$ such that

$$\begin{cases} \nabla^R f_1(x_0) = u + v, \\ \nabla^R \nabla^R f_1(x_0) = 0. \end{cases}$$

and $f_2 \in C^\infty(\mathbb{M})$ such that

$$\begin{cases} \nabla^R f_2(x_0) = 0, \\ \left(\nabla^R \nabla^R f_2(x_0)\right)_{l,j} = \frac{\nu}{2} \left(\frac{\delta_j^l + \delta_l^j}{2}\right)(x_0) v, \\ X_j Z f_2(x_0) = \frac{1}{\nu} \sum_{i=1}^{2n} \gamma_{ij}(x_0) u_i - X_j Z f_1(x_0). \end{cases}$$

where ∇^R is the Levi-Civita connection of the Riemannian metric on $\mathbb{M}$. As in [2], we can easily see the existence of such f_1. Also we can write f_2 in local coordinates $(x_1, \cdots, x_{2n}, z)$ such that

$$f_2(x,z) = \sum_{j=1}^{2n} \left(\frac{1}{\nu} \sum_{i=1}^{2n} \gamma_{ij}(x_0) u_i - X_j Z f_1(x_0) \right) x_j z + \frac{\nu}{2} \sum_{l.j=1}^{2n} \left(\frac{\delta_j^l + \delta_l^j}{2} \right)(x_0) v x_l x_j.$$

We then chose $f = f_1 + f_2$. Now we divide the rest of the proof into two parts.

(1) First we derive the bounds for $\mathbf{Ric}(\nabla_{\mathcal{H}} f) + \tau_2(f)$. From the above claim we can find a function $f \in C^\infty(\mathbb{M})$ such that (i), (ii), (iii), (iv) are satisfied with $v = 0$. Moreover, by (3.17) and (3.22) we have that

$$\Gamma_2(f) + \nu \Gamma_2^Z(f) = \mathbf{Ric}(\nabla_{\mathcal{H}} f) + \tau_2(f)$$

Hence we have that for all $\nu > 0$, $z > 0$, $w > 0$,

$$\mathbf{Ric}(\nabla_{\mathcal{H}} f)(x_0) + \tau_2(f)(x_0) \geq (\rho_1 - \frac{\kappa}{\nu}) \|u\|^2$$

where $\rho_1 = c_1 - \frac{c_2 z}{2} - \frac{c_3 w}{2}$. By letting $\nu \to \infty$, $z \to 0$, $w \to 0$, we obtain that

$$\mathbf{Ric}(\nabla_{\mathcal{H}} f)(x_0) + \tau_2(f)(x_0) \geq c_1 \|u\|^2.$$

(2) To derive the bound for $\|\tau\|^2$, we notice that the existence of the function $f \in C^\infty(\mathbb{M})$ satisfying (i), (ii), (iii), (iv) implies

$$\begin{aligned} &\Gamma_2(f) + \nu \Gamma_2^Z(f) \\ =&\mathcal{R}(f,f) - \frac{1}{\nu} \sum_{j=1}^{2n} \left(\sum_{i=1}^{2n} \gamma_{ij} X_i f \right)^2 + \nu \sum_{i,j,l=1}^{2n} \left(\frac{\delta_j^l + \delta_l^j}{2} \right) \left(\frac{w_{il}^j + w_{ij}^l}{2} \right) X_i f Z f \\ &+ \nu \sum_{l=1}^{2n} \left(\sum_{i=1}^{2n} X_i \delta_i^l - \sum_{i,k=1}^{2n} w_{ik}^k \delta_i^l + \sum_{k=1}^{2n} Z w_{lk}^k \right) X_l f Z f - \frac{\nu^2}{2} \sum_{1 \leq l < j \leq 2n} \left(\left(\frac{\delta_j^l + \delta_l^j}{2} \right) Z f \right)^2. \end{aligned}$$

Since

$$\Gamma_2(f,f) + \nu \Gamma_2^Z(f,f) \geq (\rho_1 - \frac{1}{\nu}) \|u\|^2 + (\rho_2 - \rho_3 \nu^2) \|v\|^2,$$

by comparing the coefficients of ν^2 terms we have that

$$\frac{1}{2}\sum_{1\le l<j\le 2n}\left(\left(\frac{\delta_j^l+\delta_l^j}{2}\right)Zf\right)^2 \le \frac{c_3}{2w}+\frac{\iota}{4}$$

for all $w>0$. Let $w\to\infty$ we obtain

$$\|\tau\|^2=\sum_{l,j=1}^{2n}\left(\left(\frac{\delta_j^l+\delta_l^j}{2}\right)Zf\right)^2 \le \iota.$$

□

4 Stochastic completeness and Bonnet Myers type theorem

Throughout this section we assume that L satisfies the generalized curvature dimension condition $\mathrm{CD}(\rho_1,\ \rho_2,\ \rho_3,\ \kappa,\ \infty)$ with $\rho_1\in\mathbb{R}$, $\rho_2>0$, $\rho_3>0$, $\kappa>0$. Our purpose here is to study the stochastic completeness property of the heat semigroup and the compactness properties of the manifold $\mathbb{M}$.

Let us introduce the rescaled Riemannian metric

$$g_\lambda = d\theta(\cdot, J\cdot)+\lambda^{-2}\theta^2,$$

where $\lambda>0$. The associated Laplacian Δ^λ is given by

$$\Delta^\lambda = L+\lambda^2 Z^2$$

and the associated first order bilinear form is

$$\Gamma^\lambda(f)=\Gamma(f)+\lambda^2(Zf)^2.$$

Lemma 4.1 *If there exists $\alpha,\iota\ge 0$ such that for every $f\in C^\infty(\mathbb{M})$,*

$$\langle(\nabla_Z\tau)(\nabla_{\mathcal{H}}f),\nabla_{\mathcal{H}}f\rangle \le \alpha\|\nabla_{\mathcal{H}}f\|^2, \quad \|\tau(\nabla_{\mathcal{H}}f)\|^2\le \iota\|\nabla_{\mathcal{H}}f\|^2, \tag{4.31}$$

then we have

$$\Gamma_2^\lambda(f)\ge \Gamma_2(f)+\lambda^2\Gamma_2^Z(f)-\lambda^2\left(2\iota+\alpha\right)\Gamma(f), \tag{4.32}$$

and consequently

$$\Gamma_2^\lambda(f)\ge c(\lambda)\Gamma^\lambda(f),$$

where $c(\lambda)=\min\left\{\rho_1-\frac{\kappa}{\lambda^2}+\lambda^2\left(2\iota+\alpha\right),\frac{\rho_2}{\lambda^2}-\rho_3\lambda^2\right\}$.

Proof. Some easy computations show that

$$\begin{aligned} 2\Gamma_2^\lambda(f) &= \Delta^\lambda\Gamma^\lambda(f) - 2\Gamma^\lambda(f,\Delta^\lambda(f)) \\ &= 2\Gamma_2(f) + \lambda^2\left(Z^2\Gamma(f) - 2\Gamma(f,Z^2f) + 2\Gamma_2^Z(f)\right) + 2\lambda^4(Z^2f)^2, \end{aligned}$$

and, in a local orthonormal frame,

$$Z^2\Gamma(f) - 2\Gamma(f,Z^2f) = 2\sum_{k=1}^{2n}(ZX_kf - \sum_{i=1}^{2n}\delta_i^kX_if)^2 - 2\sum_{i,j,k=1}^{2n}\delta_i^k(\delta_j^k+\delta_k^j)X_ifX_jf - 2\sum_{i,k=1}^{2n}(Z\delta_i^k)X_ifX_kf.$$

Since

$$ZX_kf = X_kZf - \sum_{i=1}^{2n}\delta_k^iX_if$$

and

$$\sum_{i=1}^{2n}\Big((\nabla_Z\tau)(X_i)\Big)fX_if = \sum_{i,k=1}^{2n}Z(\delta_i^k)X_ifX_kf + \frac{1}{2}\sum_{i,j,k=1}^{2n}(\delta_i^k\delta_j^k - \delta_k^i\delta_k^j)X_ifX_jf,$$

we can conclude that

$$\frac{1}{2}Z^2\Gamma(f) - \Gamma(f,Z^2f) = \sum_{k=1}^{2n}(X_kZf - 2\tau(X_k)f)^2 - 2\|\tau(\nabla_{\mathcal{H}}f)\|^2 - \langle(\nabla_Z\tau)(\nabla_{\mathcal{H}}f),\nabla_{\mathcal{H}}f\rangle, \tag{4.33}$$

and thus

$$\frac{1}{2}Z^2\Gamma(f) - \Gamma(f,Z^2f) \geq -2\|\tau(\nabla_{\mathcal{H}}f)\|^2 - \langle(\nabla_Z\tau)(\nabla_{\mathcal{H}}f),\nabla_{\mathcal{H}}f\rangle.$$

At the end we obtain (4.32) by plugging in (4.31). The inequality (4.32) is obtained by using the generalized curvature condition $CD(\rho_1,\, \rho_2,\, \rho_3,\, \kappa,\, \infty)$. □

This lemma has a very interesting first corollary.

Theorem 4.2 *Assume that there exists $\alpha,\iota \geq 0$ such that for every $f \in C^\infty(\mathbb{M})$,*

$$\langle(\nabla_Z\tau)(\nabla_{\mathcal{H}}f),\nabla_{\mathcal{H}}f\rangle \leq \alpha\|\nabla_{\mathcal{H}}f\|^2, \quad \|\tau(\nabla_{\mathcal{H}}f)\|^2 \leq \iota\|\nabla_{\mathcal{H}}f\|^2,$$

and moreover that $\rho_1 > \sqrt{\frac{\rho_3}{\rho_2}}\kappa + \sqrt{\frac{\rho_2}{\rho_3}}(2\iota+\alpha)$, then the manifold $\mathbb{M}$ is compact.

Proof. If $\rho_1 > \sqrt{\frac{\rho_3}{\rho_2}}\kappa + \sqrt{\frac{\rho_2}{\rho_3}}(2\iota+\alpha)$, then we can chose $\lambda > 0$ such that $c(\lambda) > 0$. It implies that the Ricci curvature of the Riemannian metric g^λ is bounded from below by a positive number and thus $\mathbb{M}$ is compact from the classical Bonnet-Myers theorem. □

Remark 4.3 *In the Sasakian case, $\alpha = \iota = \rho_3 = 0$ and we recover the result from [2]. However, in [2] the compactness result came with an upper bound for the Carnot-Carathéodory diameter of $\mathbb{M}$.*

A second corollary is the following volume estimate of the metric balls and the stochastic completeness of the heat semigroup. Let us first remind that the distance d associated to L is defined by:

$$d(x,y) = \sup \{ f(x) - f(y), f \in C^\infty(\mathbb{M}), \|\Gamma(f)\|_\infty \le 1 \}.$$

Theorem 4.4 *Assume that there exists $\alpha, \iota \ge 0$ such that for every $f \in C^\infty(\mathbb{M})$,*

$$\langle (\nabla_Z \tau)(\nabla_\mathcal{H} f), \nabla_\mathcal{H} f \rangle \le \alpha \|\nabla_\mathcal{H} f\|^2, \quad \|\tau(\nabla_\mathcal{H} f)\|^2 \le \iota \|\nabla_\mathcal{H} f\|^2.$$

There exist constants $C_1 \ge 0$ and $C_2 \ge 0$ such that for every $x \in \mathbb{M}$ and every $r \ge 0$

$$\mu(B(x,r)) \le C_1 e^{C_2 r}. \tag{4.34}$$

As a consequence, the heat semigroup P_t generated by the sub-Laplacian is stochastically complete, that is for every $t \ge 0$, $P_t 1 = 1$.

Proof. Let $B_\lambda(x,r)$ denote the g^λ Riemannian ball in $\mathbb{M}$ centered at x with radius r. It is easy to see that

$$B(x,r) \subset B_\lambda(x,r).$$

By Lemma 4.1, the Ricci curvature of the Riemannian metric g^λ is bounded from below. From the Riemannian volume comparison theorem, we deduce then that $\mu(B(x,r)) \le C_1 e^{C_2 r}$. As a consequence, we conclude that for every $x \in \mathbb{M}$,

$$\int_0^\infty \frac{r dr}{\log \mu(B(x,r))} = \infty.$$

Thanks to a theorem by K.T. Sturm [23], we deduce that P_t is stochastically complete.□

5 Gradient bounds for the heat semigroup and spectral gap estimates

The previous section has shown how to deduce some interesting geometric consequences of the generalized curvature dimension condition. However an additional bound is required on the tensor $\nabla_Z \tau$ and the techniques are not intrinsically associated to L in the sense that we introduced the rescaled Riemannian metric g^λ and used results from Riemannian geometry. In this Section, we develop tools to exploit in an intrinsic way the generalized curvature dimension inequality. These methods rely on the study of gradient bounds for the subelliptic heat semigroup.

In the whole section, we assume again that the sub-Laplacian of L satisfies the generalized $CD(\rho_1, \rho_2, \rho_3, \kappa, \infty)$ for some constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $\rho_3 > 0$, $\kappa > 0$. Let $P_t = e^{tL}$ be the associated heat semigroup.
In order to use heat semigroup gradient bounds techniques, we will need the following hypothesis throughout this section.

Hypothesis 5.1 *The semigroup P_t is stochastically complete, i.e., for $t > 0$,*

$$P_t 1 = 1,$$

and for all $f \in C_0^\infty(\mathbb{M})$ and $T \geq 0$, one has

$$\sup_{t \in [0,T]} \|\Gamma(P_t f)\|_\infty + \|\Gamma^Z(P_t f)\|_\infty < +\infty.$$

The *raison d'être* of this hypothesis is the following theorem that was proved in [2].

Theorem 5.2 *Assume that Hypothesis 5.1 is satisfied. Let $T > 0$. Suppose that $u, v : \mathbb{M} \times [0,T] \to \mathbb{R}$ are smooth functions such that $\sup_{t \in [0,T]} \|u(\cdot,t)\|_\infty < \infty$ and $\sup_{t \in [0,T]} \|v(\cdot,t)\|_\infty < \infty$. Suppose*

$$Lu + \frac{\partial u}{\partial t} \geq v$$

on $\mathbb{M} \times [0,T]$. Then for all $x \in \mathbb{M}$,

$$P_T(u(\cdot,T))(x) \geq u(x,0) + \int_0^T P_s(v(\cdot,s))(x) ds.$$

The Hypothesis 5.1 is not very strong. It is obviously satisfied if $\mathbb{M}$ is compact. In the non compact case, a general criterion is given in the Appendix. From now on, in this section, we assume that that Hypothesis 5.1 is satisfied.

Proposition 5.3 *Let us assume $\rho_1 - \frac{\kappa\sqrt{\rho_3}}{\sqrt{\rho_2}} \geq 0$. For $f \in C_0^\infty(\mathbb{M})$ and $t \geq 0$, we have*

$$\Gamma(P_t f) + \frac{\sigma + \sqrt{\sigma^2 + 16\rho_2\rho_3}}{4\rho_2} \Gamma^Z(P_t f) \leq e^{-\sigma t} \left(P_t(\Gamma(f)) + \frac{\sigma + \sqrt{\sigma^2 + 16\rho_2\rho_3}}{4\rho_2} P_t(\Gamma^Z(f)) \right)$$

where $\sigma = \frac{2\rho_1\rho_2 - 2\kappa\sqrt{\rho_2\rho_3}}{(\rho_2+\kappa)}$.

Proof. Let us fix $t > 0$ once time for all in the following proof. For $0 < s < t$, let

$$\phi_1(x,s) = \Gamma(P_{t-s} f)(x),$$
$$\phi_2(x,s) = \Gamma^Z(P_{t-s} f)(x),$$

be defined on $\mathbb{M} \times [0,t]$. A simple computation shows that

$$L\phi_1 + \frac{\partial \phi_1}{\partial s} = 2\Gamma_2(P_{t-s}f),$$
$$L\phi_2 + \frac{\partial \phi_2}{\partial s} = 2\Gamma_2^Z(P_{t-s}f),$$

Now consider the function

$$\phi(x,s) = a(s)\phi_1(x,s) + b(s)\phi_2(x,s)$$

Applying the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \rho_3, \kappa, \infty)$, one obtains

$$\begin{aligned} L\phi + \frac{\partial \phi}{\partial s} &= a'\Gamma(P_{t-s}f) + b'\Gamma^Z(P_{t-s}f) + 2a\Gamma_2(P_{t-s}f) + 2b\Gamma_2^Z(P_{t-s}f) \\ \geq \quad & \left(a' + 2\rho_1 a - 2\kappa \frac{a^2}{b}\right)\Gamma(P_{t-s}f) + \left(b' + 2\rho_2 a - 2\rho_3 \frac{b^2}{a}\right)\Gamma^Z(P_{t-s}f). \end{aligned} \tag{5.35}$$

Let us chose

$$b(s) = e^{\frac{-2\rho_1\rho_2 + 2\kappa\sqrt{\rho_2\rho_3}}{(\rho_2+\kappa)}s},$$

and

$$a(s) = \frac{\sigma + \sqrt{\sigma^2 + 16\rho_2\rho_3}}{4\rho_2} b(s),$$

where $\sigma = \frac{2\rho_1\rho_2 - 2\kappa\sqrt{\rho_2\rho_3}}{(\rho_2+\kappa)}$, and denote $\delta = \frac{\sigma + \sqrt{\sigma^2+16\rho_2\rho_3}}{4\rho_2}$. It is easy to observe that

$$b'(s) = -\sigma b(s), \quad a'(s) = -\sigma a(s) = -\sigma\delta b(s).$$

We now claim that $a(s)$, $b(s)$ satisfy

$$a' + 2a\rho_1 - 2\kappa \frac{a^2}{b} \geq 0, \tag{5.36}$$
$$b' + 2a\rho_2 - 2\rho_3 \frac{b^2}{a} = 0. \tag{5.37}$$

Indeed, (5.37) writes as

$$-\sigma\delta + 2\delta^2\rho_2 - 2\rho_3 = 0,$$

and follows immediately by the expressions of δ. To see (5.36), similarly, we only need to prove that

$$-\delta\sigma + 2\rho_1\delta - 2\kappa\delta^2 \geq 0,$$

which is equivalent to prove

$$2\rho_1 \geq 2\kappa\delta + \sigma,$$

We can obtain it by observing that

$$\kappa\sqrt{\sigma^2 + 16\rho_2\rho_3} \leq 4\kappa\sqrt{\rho_2\rho_3} + \kappa\sigma,$$

thus we have claim proved. Plug (5.36) and (5.37) into (5.35), we get

$$L\phi + \frac{\partial \phi}{\partial s} \geq 0$$

and by the comparison result of Theorem 5.2, we have that

$$P_T(\phi(\cdot,t))(x) \geq \phi(x,0).$$

We complete the proof by realizing that

$$\phi(x,0) = a(0)\Gamma(P_t f)(x) + b(0)\Gamma^Z(P_t f)(x),$$

and

$$P_t(\phi(\cdot,t))(x) = a(t)P_t(\Gamma(f))(x) + b(t)P_t(\Gamma^Z(f))(x).$$

□

A direct application of the above inequality is the fact $\rho_1 - \frac{\kappa\sqrt{\rho_3}}{\sqrt{\rho_2}} > 0$ implies that the invariant measure is finite.

Corollary 5.4 *If* $\rho_1 - \frac{\kappa\sqrt{\rho_3}}{\sqrt{\rho_2}} > 0$ *then* $\mathbb{M}$ *has a finite volume, i.e.,*

$$\mu(\mathbb{M}) < +\infty.$$

Proof. Let $f, g \in C_0^\infty(\mathbb{M})$, and write

$$\int_{\mathbb{M}} (P_t f - f) g d\mu = \int_{\mathbb{M}} \int_0^t \frac{\partial(P_s f)}{\partial s} g ds d\mu = \int_0^t \int_{\mathbb{M}} (LP_s f) g d\mu ds = -\int_0^t \int_{\mathbb{M}} \Gamma(P_s f, g) d\mu ds$$

By Cauchy-Schwartz inequality, we have

$$\left| \int_{\mathbb{M}} (P_t f - f) g d\mu \right| \leq \int_0^t \int_{\mathbb{M}} \left(\Gamma(P_s f)^{\frac{1}{2}} \Gamma(g)^{\frac{1}{2}} \right) d\mu ds.$$

Applying Proposition 5.3, we obtain then

$$\left| \int_{\mathbb{M}} (P_t f - f) g d\mu \right| \leq \left(\int_0^t e^{\frac{\rho_1\rho_2 - 2\kappa\sqrt{\rho_2\rho_3}}{(\rho_2+\kappa)} s} ds \int_{\mathbb{M}} \Gamma(g)^{\frac{1}{2}} d\mu \right) \sqrt{\|\Gamma(f)\|_\infty + \frac{\sigma + \sqrt{\sigma^2 + 16\rho_2\rho_3}}{4\rho_2} \|\Gamma^Z(f)\|_\infty},$$

where $\sigma = \frac{2\rho_1\rho_2 - 2\kappa\sqrt{\rho_2\rho_3}}{(\rho_2+\kappa)}$.
Moreover, from the spectral theorem we know that $P_t f$ converges to $P_\infty f$ in $L^2(\mathbb{M})$ and $LP_\infty f = 0$, where $P_\infty f$ is in the domain of L. Hence $\Gamma(P_\infty f) = 0$, which implies that $P_\infty f$ is a constant.
We then prove the measure is finite by contradiction. Assume $\mu(\mathbb{M}) = +\infty$, then we have $P_\infty f = 0$, thus when $t \to +\infty$,

$$\left| \int_{\mathbb{M}} f g d\mu \right| \leq \left(\int_0^{+\infty} e^{\frac{\rho_1\rho_2 - 2\kappa\sqrt{\rho_2\rho_3}}{(\rho_2+\kappa)} s} ds \int_{\mathbb{M}} \Gamma(g)^{\frac{1}{2}} d\mu \right) \sqrt{\|\Gamma(f)\|_\infty + \frac{\sigma + \sqrt{\sigma^2 + 16\rho_2\rho_3}}{4\rho_2} \|\Gamma^Z(f)\|_\infty}.$$

Let $g \geq 0$, $g \neq 0$, and we chose for f an increasing sequence $\{h_k\} \in C_0^\infty(\mathbb{M})$ such that $h_k \nearrow 1$ on $\mathbb{M}$ and

$$\|\Gamma(h_k)\|_\infty + \|\Gamma^Z(h_k)\|_\infty \to_{k\to+\infty} 0.$$

By letting $k \to +\infty$, we obtain

$$\int_{\mathbb{M}} g d\mu \leq 0,$$

which is a contradiction. Hence $\mu(\mathbb{M}) < +\infty$. □

Another corollary is the following Poincaré inequality.

Corollary 5.5 *If $\rho_1 - \frac{\kappa\sqrt{\rho_3}}{\sqrt{\rho_2}} > 0$, then for all $f \in C_0^\infty(\mathbb{M})$,*

$$\int_{\mathbb{M}} f^2 d\mu - \left(\int_{\mathbb{M}} f d\mu\right)^2 \leq \frac{\rho_2 + \kappa}{\rho_1\rho_2 - \kappa\sqrt{\rho_2\rho_3}} \int_{\mathbb{M}} \Gamma(f) d\mu. \tag{5.38}$$

Proof. By proposition 5.3, we have

$$\begin{aligned}\int_{\mathbb{M}} \Gamma(P_t f) d\mu &\leq e^{\frac{2\rho_1\rho_2 - 2\kappa\sqrt{\rho_2\rho_3}}{(\rho_2+\kappa)}t} \int_{\mathbb{M}} \left(P_t(\Gamma(f)) + \frac{\sigma + \sqrt{\sigma^2 + 16\rho_2\rho_3}}{4\rho_2} P_t(\Gamma^Z(f))\right) d\mu \\ &\leq e^{\frac{2\rho_1\rho_2 - 2\kappa\sqrt{\rho_2\rho_3}}{(\rho_2+\kappa)}t} \int_{\mathbb{M}} \left(\Gamma(f) + \frac{\sigma + \sqrt{\sigma^2 + 16\rho_2\rho_3}}{4\rho_2} \Gamma^Z(f)\right) d\mu, \end{aligned} \tag{5.39}$$

where the last inequality is due to the contractivity of P_t. Let dE_λ be the spectral resolution of $-L$. Then by the spectral theorem we have

$$\int_{\mathbb{M}} \Gamma(P_t f) d\mu = \int_0^{+\infty} \lambda e^{-2\lambda t} dE_\lambda(f) \tag{5.40}$$

and

$$\int_{\mathbb{M}} \Gamma(f) d\mu = \int_0^{+\infty} \lambda dE_\lambda(f).$$

Thus for $0 < s < t$, by Hölder inequality,

$$\int_{\mathbb{M}} \Gamma(P_s f) d\mu = \int_0^{+\infty} \lambda e^{-2\lambda s} dE_\lambda(f) \leq \left(\int_0^\infty \lambda e^{-2\lambda t} dE_\lambda(f)\right)^{\frac{s}{t}} \left(\int_0^\infty \lambda dE_\lambda(f)\right)^{\frac{t-s}{t}}. \tag{5.41}$$

We denote $C(f) = \int_{\mathbb{M}} \left(\Gamma(f) + \frac{\sigma + \sqrt{\sigma^2+16\rho_2\rho_3}}{4\rho_2}\Gamma^Z(f)\right) d\mu$, then by (5.39), (5.40) and (5.41) we have

$$\int_{\mathbb{M}} \Gamma(P_s f) d\mu \leq e^{\frac{2\rho_1\rho_2 - 2\kappa\sqrt{\rho_2\rho_3}}{(\rho_2+\kappa)}s} C(f)^{\frac{s}{t}} \left(\int_{\mathbb{M}} \Gamma(f) d\mu\right)^{\frac{t-s}{t}}.$$

By letting $t \to +\infty$, we obtain

$$\int_{\mathbb{M}} \Gamma(P_s f) d\mu \leq e^{\frac{2\rho_1\rho_2 - 2\kappa\sqrt{\rho_2\rho_3}}{(\rho_2+\kappa)}s} \int_{\mathbb{M}} \Gamma(f) d\mu.$$

At the end, we obtain the desired Poincaré inequality by observing

$$\int_{\mathbb{M}} f^2 d\mu - \left(\int_{\mathbb{M}} f d\mu \right)^2 = -\int_0^\infty \frac{\partial}{\partial s} \int_{\mathbb{M}} (P_s f)^2 d\mu ds = \int_{\mathbb{M}} \Gamma(P_s f) d\mu.$$

□

This result naturally raises the conjecture that if $\rho_1 - \frac{\kappa\sqrt{\rho_3}}{\sqrt{\rho_2}} > 0$, then $\mathbb{M}$ is compact. This would be a stronger result than Theorem 4.2.

6 Appendix: Gradient bounds by stochastic analysis

The goal of the section is to study some general conditions ensuring that Hypothesis 5.1 is true.

Let $\mathbb{M}$ be a $n+m$ dimensional smooth manifold. We assume given $n+m$ smooth vector fields $\{X_1, \cdots, X_{n+m}\}$ on $\mathbb{M}$ such that for every $x \in \mathbb{M}$, $\{X_1(x), \cdots, X_{n+m}(x)\}$ is a basis of $T_x\mathbb{M}$. This global basis of vector fields induce on $\mathbb{M}$ a Riemannian metric g that we assume to be complete. There exist smooth functions $\omega_{ij}^k : \mathbb{M} \to \mathbb{R}$, $i, j, k = 1, \cdots, n+m$, such that:

$$[X_i, X_j] = \sum_{k=1}^{n+m} \omega_{ij}^k X_k.$$

We assume that the vector fields $\{X_1, \cdots, X_n\}$ satisfy the Hörmander's bracket generating condition.

Let us consider the symmetric and subelliptic operator

$$L = -\frac{1}{2} \sum_{i=1}^n X_i^* X_i,$$

where $X_i^* = -X_i + \mathbf{div} X_i$ is the formal adjoint of X_i with respect to the Riemannian measure μ. By using a similar argument as in the proof of Lemma 2.1, it is seen that the assumed completeness of g implies that L is essentially self-adjoint on the space $C_0^\infty(\mathbb{M})$. As a consequence, L is the generator of sub-Markov semigroup $(P_t)_{t \ge 0}$. Let us observe that L can also be written as

$$L = X_0 + \frac{1}{2} \sum_{k=1}^n X_k^2,$$

where $X_0 = -\frac{1}{2} \sum_{i=1}^n (\mathbf{div} X_i) X_i = -\frac{1}{2} \sum_{i=1}^n \sum_{k=1}^{n+m} \omega_{ik}^k X_i$. We thus can find some smooth functions ω_{0i}^k's such that

$$[X_0, X_i] = \sum_{k=1}^{n+m} \omega_{0i}^k X_k.$$

Let now $(B_t)_{t \ge 0}$ be a n-dimensional Brownian motion.

If we consider the stochastic differential equation on $\mathbb{M}$,

$$dY_t^x = \sum_{k=0}^{n} X_k(Y_t^x) \circ dB_t^k, \quad Y_0^x = x,$$

with the notation $B_t^0 = t$, it has a unique solution defined up to and explosion time $\mathbf{e}(x)$. If f is a bounded Borel function on $\mathbb{M}$, we then have the representation

$$P_t f(x) = \mathbb{E}\left(f(Y_t^x) 1_{t<\mathbf{e}(x)} \right).$$

Our goal is to prove the following theorem:

Theorem 6.1 *Let us assume that the functions ω_{ij}^k, $X_l \omega_{ij}^k$, $i,j,k,l = 1, \cdots, n+m$ are bounded, then the semigroup P_t is stochastically complete and there exist constants $C_1, C_2 \geq 0$ such that for every $f \in C_0^\infty(\mathbb{M})$, $t \geq 0$ and $x \in \mathbb{M}$*

$$\sum_{k=1}^{n+m} (X_k P_t f)^2(x) \leq C_1 e^{C_2 t} \left(\sum_{k=1}^{n+m} \| X_k f \|_\infty^2 \right).$$

Proof. We adapt some ideas from Kusuoka [15]. Let $x, y \in \mathbb{M}$ and let $\mathcal{O}$ be a bounded open set that contains the Riemannian geodesic connecting x to y. Let $R > 0$ such that the ball $B(x, R)$ with center x and radius R contains $\mathcal{O}$. We denote

$$T_R = \inf_{z \in \overline{\mathcal{O}}} \inf\{ t \geq 0, Y_t^z \notin B(x,R) \}.$$

Let us then consider for $f \in C_0^\infty(\mathbb{M})$, and $z \in \mathcal{O}$,

$$P_t^R f(z) = \mathbb{E}\left(f(Y_{t\wedge T_R}^z) \right).$$

By using the chain rule, and the triangle inequality, we see that for $z \in \mathcal{O}$,

$$\sum_{k=1}^{n+m} (X_k P_t^R f)^2(z) \leq \mathbb{E}\left(\| J_{t\wedge T_R}^*(z) \nabla f(Y_{t\wedge T_R}^z) \| \right)^2 \leq \mathbb{E}\left(\| J_{t\wedge T_R}^*(z) \| \right)^2 \left(\sum_{k=1}^{n+m} \| X_k f \|_\infty^2 \right).$$

where $J_t(z) = \frac{\partial Y_t^z}{\partial z}, t < T_R$, is the first variation process of the stochastic differential equation. We thus want to find a bound for $\mathbb{E}\left(\| J_{t\wedge T_R}^*(z) \| \right)$ that does not depend on R and z. Since $\{X_1, \cdots, X_{n+m}\}$ form a basis at each point, we can find processes $\beta_i^k(t,z)$, $k = 1, \cdots, m+n$, $i = 1, \cdots n$ such that for $t < T_R$,

$$J_t^{-1}(X_i(Y_t^z)) = \sum_{k=1}^{m+n} \beta_i^k(t,z) X_k(z).$$

By using the chain rule, we have for $t < T_R$,

$$
\begin{aligned}
dJ_t^{-1}(X_i(Y_t^z)) &= \sum_{k=0}^{n} J_t^{-1}([X_k, X_i](Y_t^z)) \circ dB_t^k \\
&= \sum_{k=0}^{n} \sum_{l=1}^{m+n} \omega_{ki}^l(Y_t^z) J_t^{-1}(X_k(Y_t^z)) \circ dB_t^k.
\end{aligned}
$$

As a consequence the matrix valued process $\beta(t,z)$, $t < T_R$ solves the matrix stochastic differential equation,

$$
d\beta(t,z) = \sum_{k=0}^{n} \omega_k(Y_t^z)\beta(t,z) \circ dB_t^k.
$$

The inverse matrix process $\alpha(t,z) = \beta(t,z)^{-1}$ will then solve the linear stochastic differential equation for $t < T_R$,

$$
d\alpha(t,z) = -\sum_{k=0}^{n} \alpha(t,z)\omega_k(Y_t^z) \circ dB_t^k.
$$

From our assumption, all the coefficients of the equation are bounded. We therefore obtain a bound $\mathbb{E}(\|\alpha(t,z)\|) \le C_1 e^{C_2 t}$, where C_1, C_2 are independent from R and z. As a conclusion, we get

$$
\sum_{k=1}^{n+m} (X_i P_t^R f)^2(z) \le C_1 e^{C_2 t} \left(\sum_{k=1}^{n+m} \|X_k f\|_\infty^2 \right).
$$

By integrating the inequality over the geodesic between x and y, we obtain

$$
|(P_t^R f)(x) - (P_t^R f)(y)|^2 \le C_1 e^{C_2 t} \left(\sum_{k=1}^{n+m} \|X_k f\|_\infty^2 \right) d(x,y)^2.
$$

We can then let $R \to \infty$ to conclude

$$
|(P_t f)(x) - (P_t f)(y)|^2 \le C_1 e^{C_2 t} \left(\sum_{k=1}^{n+m} \|X_k f\|_\infty^2 \right) d(x,y)^2.
$$

Since this is true for every $x, y \in \mathbb{M}$, we conclude

$$
\sum_{k=1}^{n+m} (X_k P_t f)^2(x) \le C_1 e^{C_2 t} \left(\sum_{k=1}^{n+m} \|X_k f\|_\infty^2 \right).
$$

We now prove the stochastic completeness. Let $f, g \in C_0^\infty(\mathbb{M})$, we have

$$
\int_{\mathbb{M}} (P_t f - f) g d\mu = \int_0^t \int_{\mathbb{M}} \left(\frac{\partial}{\partial s} P_s f \right) g d\mu ds = \int_0^t \int_{\mathbb{M}} (L P_s f)\, g d\mu ds = -\int_0^t \int_{\mathbb{M}} \Gamma(P_s f, g) d\mu ds.
$$

By means of Cauchy-Schwarz inequality we find

$$\left|\int_{\mathbb{M}}(P_t f - f) g d\mu\right| \le \left(\int_0^t C_1 e^{C_2 s} ds\right) \|\nabla f\|_\infty \int_{\mathbb{M}} \Gamma(g)^{\frac{1}{2}} d\mu. \tag{6.42}$$

We now apply (6.42) with $f = h_k$, where h_k is a sequence such that $h_k \nearrow 1$, $h_k \ge 0$ and $\sum_{k=1}^{n+m} \|X_k h_l\|_\infty^2 \to 0$ when $l \to \infty$.
By Beppo Levi's monotone convergence theorem we have $P_t h_k(x) \nearrow P_t 1(x)$ for every $x \in \mathbb{M}$. We conclude that the left-hand side of (6.42) converges to $\int_{\mathbb{M}}(P_t 1 - 1) g d\mu$. Since the right-hand side converges to zero, we reach the conclusion

$$\int_{\mathbb{M}} (P_t 1 - 1) g d\mu = 0, \quad g \in C_0^\infty(\mathbb{M}).$$

It follows that $P_t 1 = 1$. □

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